A NOTE ON THE NEUMAN-SÁNDOR MEAN

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Abstract. In this article, we present the best possible upper and lower bounds for the Neuman-Sándor mean in terms of the geometric combinations of harmonic and quadratic means, geometric and quadratic means, harmonic and contra-harmonic means, and geometric and contra-harmonic means.

1. Introduction

For $a, b > 0$ with $a \neq b$ the Neuman-Sándor mean $M(a, b)$ [1] is defined by

$$M(a,b) = \frac{a-b}{2\sinh^{-1}\left(\frac{a-b}{a+b}\right)}, \tag{1.1}$$

where $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a, b)$ can be found in the literature [1, 2].

Let $H(a, b) = 2ab/(a + b)$, $G(a, b) = \sqrt{ab}$, $L(a, b) = (b - a)/(\log b - \log a)$, $P(a, b) = (a - b)/(4 \arctan \sqrt{a/b} - \pi)$, $A(a, b) = (a + b)/2$, $T(a, b) = (a - b)/[2 \arctan((a - b)/(a + b))]$, $Q(a, b) = \sqrt{(a^2 + b^2)/2}$ and $C(a, b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic and contra-harmonic means of a and b, respectively. Then it is well-known that the inequalities

$$H(a,b) < G(a,b) < L(a,b) < P(a,b) < A(a,b) < M(a,b) < T(a,b) < Q(a,b) < C(a,b)$$

hold true for $a, b > 0$ with $a \neq b$.

Neuman and Sándor [1, 2] established that

$$A(a,b) < M(a,b) < T(a,b),$$

$$P(a,b)M(a,b) < A^2(a,b),$$

$$A(a,b)T(a,b) < M^2(a,b) < [A^2(a,b) + T^2(a,b)]/2$$

hold for all $a, b > 0$ with $a \neq b$.

Let $0 < a, b < 1/2$ with $a \neq b$, $a' = 1 - a$ and $b' = 1 - b$. Then the following Ky Fan inequalities

$$\frac{G(a,b)}{G(a',b')} < \frac{L(a,b)}{L(a',b')} < \frac{P(a,b)}{P(a',b')} < \frac{A(a,b)}{A(a',b')} < \frac{M(a,b)}{M(a',b')} < \frac{T(a,b)}{T(a',b')}$$

were presented in [1].

Li et al. [3] showed that the double inequality $L_{p_0}(a, b) < M(a, b) < L_2(a, b)$ holds for all $a, b > 0$ with $a \neq b$, where $L_p(a, b) = \left[(b^{p+1} - a^{p+1})/((p + 1)(b - a))\right]^{1/p}$ $(p \neq -1, 0)$, $L_0 = 1/e(b^b/a^a)^{1/(b-a)}$ and $L_{-1}(a, b) = (b - a)/(\log b - \log a)$ be the $p-$th generalized logarithmic mean of a and b, and $p_0 = 1.843\cdots$ is the unique solution of the equation $(p + 1)^{1/p} = 2\log(1 + \sqrt{2})$.

In [4], Neuman proved that the double inequalities

$$Q^{\alpha}(a,b)A^{1-\alpha}(a,b) < M(a,b) < Q^{\beta}(a,b)A^{1-\beta}(a,b)$$

and

$$C^{\lambda}(a,b)A^{1-\lambda}(a,b) < M(a,b) < C^{\mu}(a,b)A^{1-\mu}(a,b)$$

2000 *Mathematics Subject Classification.* 26E60.

Key words and phrases. Neuman-Sándor mean, harmonic mean, geometric mean, quadratic mean, contra-harmonic mean.

This research was supported by the Natural Science Foundation of China under Grants 11071069 and 11171307, and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant T200924.

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq 1/3$, $\beta\geq 2\left(\log(2+\sqrt{2})-\log 3\right)/\log 2=0.373\cdots$, $\lambda\leq 1/6$ and $\mu\geq\left(\log(2+\sqrt{2})-\log 3\right)/\log 2=0.186\cdots$.

The main purpose of this paper is to find the least values α_1, α_2, α_3, α_4, and the greatest values β_1, β_2, β_3, β_4 such that the double inequalities

$$H^{\alpha_1}(a,b)Q^{1-\alpha_1}(a,b)<M(a,b)<H^{\beta_1}(a,b)Q^{1-\beta_1}(a,b),$$
$$G^{\alpha_2}(a,b)Q^{1-\alpha_2}(a,b)<M(a,b)<G^{\beta_2}(a,b)Q^{1-\beta_2}(a,b),$$
$$H^{\alpha_3}(a,b)C^{1-\alpha_3}(a,b)<M(a,b)<H^{\beta_3}(a,b)C^{1-\beta_3}(a,b)$$

and

$$G^{\alpha_4}(a,b)C^{1-\alpha_4}(a,b)<M(a,b)<G^{\beta_4}(a,b)C^{1-\beta_4}(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

2. Lemmas

In order to establish our main results we need four lemmas, which we present in this section.

Lemma 2.1. *(See [5], Theorem 1.25). For $-\infty<a<b<\infty$, let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b), let $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)}\quad\text{and}\quad\frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2. *(See [6], Lemma 1.1). Suppose that the power series $f(x)=\sum\limits_{n=0}^{\infty}a_nx^n$ and $g(x)=\sum\limits_{n=0}^{\infty}b_nx^n$ have the radius of convergence $r>0$ and $b_n>0$ for all $n\in\{0,1,2,\cdots\}$. Let $h(x)=f(x)/g(x)$, then*

(1) If the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0,r)$;

(2) If the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing) for $0<n\leq n_0$ and (strictly) decreasing (increasing) for $n>n_0$, then there exists $x_0\in(0,r)$ such that $h(x)$ is (strictly) increasing (decreasing) on $(0,x_0)$ and (strictly) decreasing (increasing) on (x_0,r).

Lemma 2.3. *Let*

$$\phi(t)=\frac{[3-\cosh(2t)][\sinh(2t)-2t]}{2t\sinh^2(t)[5+\cosh(2t)]},\tag{2.1}$$

then $\phi(t)$ is strictly decreasing in $(0,\log(1+\sqrt{2}))$, where $\sinh(t)=(e^t-e^{-t})/2$ and $\cosh(t)=(e^t+e^{-t})/2$ are respectively the hyperbolic sine and cosine functions.

Proof. Let us denote by $\phi_1(t)$ and $\phi_2(t)$ respectively the numerator and denominator of (2.1) expand the factor to obtain

$$\phi_1(t)=3\sinh(2t)-6t+2t\cosh(2t)-\frac{1}{2}\sinh(4t),\tag{2.2}$$

$$\phi_2(t)=\frac{t}{2}\left[8\cosh(2t)+\cosh(4t)-9\right].\tag{2.3}$$

Using the power series $\sinh(t)=\sum_{n=0}^{\infty}t^{2n+1}/(2n+1)!$ and $\cosh(t)=\sum_{n=0}^{\infty}t^{2n}/(2n)!$, we can express (2.2) and (2.3) as follows

$$\phi_1(t)=\sum_{n=1}^{\infty}\frac{2^{2n+1}(2n+4-2^{2n})}{(2n+1)!}t^{2n+1}=t^3\sum_{n=0}^{\infty}\frac{2^{2n+4}(n+3-2^{2n+1})}{(2n+3)!}t^{2n},\tag{2.4}$$

$$\phi_2(t)=\sum_{n=1}^{\infty}\frac{2^{2n}(4+2^{2n-1})}{(2n)!}t^{2n+1}=t^3\sum_{n=0}^{\infty}\frac{2^{2n+4}(1+2^{2n-1})}{(2n+2)!}t^{2n}.\tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\phi(t)=\frac{\sum\limits_{n=0}^{\infty}a_nt^{2n}}{\sum\limits_{n=0}^{\infty}b_nt^{2n}}\tag{2.6}$$

with $a_n = 2^{2n+4}(n+3-2^{2n+1})/(2n+3)!$ and $b_n = 2^{2n+4}(1+2^{2n-1})/(2n+2)!$.

Let $c_n = a_n/b_n$, then simple computations lead to

$$c_n = \frac{(n+3)-2^{2n+1}}{(2n+3)(1+2^{2n-1})},$$

$$c_0 = \frac{2}{9} > c_1 = -\frac{4}{15} > c_2 = -\frac{3}{7} < c_3 = -\frac{122}{297}, \tag{2.7}$$

$$\begin{aligned} c_{n+1} - c_n =& \frac{2^{4n+3} - (6n^2+57n+76)2^{2n-1} - 3}{(2n+3)(2n+5)(1+2^{2n-1})(1+2^{2n+1})} \\ =& \frac{[2(4^n-38)+6(4^n-n^2)+(128\times 4^{n-2}-57n)]2^{2n-1}-3}{(2n+3)(2n+5)(1+2^{2n-1})(1+2^{2n+1})} > 0 \end{aligned} \tag{2.8}$$

for all $n > 2$.

Inequalities (2.7) and (2.8) implies that the sequence $\{a_n/b_n\}$ is strictly decreasing in $0 < n \le 2$ and strictly increasing for $n > 2$, then from (2.6) and Lemma 2.2(2) we know that there exists $t_0 > 0$ such that $\phi(t)$ is strictly decreasing on $(0, t_0)$ and strictly increasing in (t_0, ∞).

For convenience, let us denote $t^\star = \log(1+\sqrt{2}) = 0.881\cdots$, then we have

$$\sinh(t^*) = 1, \quad \sinh(2t^*) = 2\sqrt{2}, \quad \sinh(3t^*) = 7, \tag{2.9}$$

$$\cosh(t^*) = \sqrt{2}, \quad \cosh(2t^*) = 3, \quad \cosh(3t^*) = 5\sqrt{2}. \tag{2.10}$$

Differentiating (2.1) yields

$$\phi'(t) = \frac{{\phi_1}'(t)\phi_2(t) - \phi_1(t){\phi_2}'(t)}{{\phi_2}^2(t)}, \tag{2.11}$$

where

$$\phi_1'(t) = 8\sinh(t)[t\cosh(t) - 2\sinh^3(t)], \tag{2.12}$$

$$\phi_2'(t) = \sinh(t)[20t\cosh(t) + 4t\cosh(3t) + 9\sinh(t) + \sinh(3t)]. \tag{2.13}$$

From (2.2) and (2.3) together with (2.9)-(2.13) we get

$$\phi'(t^*) = -\frac{\sqrt{2}-t^*}{\sqrt{2}t^*} < 0. \tag{2.14}$$

It follows from the piecewise monotonicity of $\phi(t)$ and (2.14) that $t_0 > t^*$. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Let $p \in [0,1)$, and*

$$\varphi_p(t) = \log(1+x^2) - \log\frac{x}{\sinh^{-1}(x)} + p\left[\frac{1}{2}\log(1-x^2) - \log(1+x^2)\right]. \tag{2.15}$$

Then $\varphi_{5/9}(x) < 0$ and $\varphi_0(x) > 0$ for all $x \in (0,1)$.

Proof. From (2.15) one has

$$\varphi_p(0^+) = 0, \tag{2.16}$$

$$\varphi_p'(x) = \frac{\phi_p(x)}{x(1-x^4)\sqrt{1+x^2}\sinh^{-1}(x)}, \tag{2.17}$$

where

$$\phi_p(x) = x - x^5 - [1+(3p-2)x^2+(1-p)x^4]\sqrt{1+x^2}\sinh^{-1}(x). \tag{2.18}$$

We divide the proof into two cases.

Case 1 $p = 5/9$. Then (2.18) leads to

$$\phi_{5/9}(0) = 0, \tag{2.19}$$

$$\phi_{5/9}'(x) = -\frac{xf(x)}{9\sqrt{1+x^2}}, \tag{2.20}$$

where

$$f(x) = x(49x^2-3)\sqrt{1+x^2} + (3+7x^2+20x^4)\sinh^{-1}(x), \tag{2.21}$$

$$f(0) = 0. \tag{2.22}$$

Differentiating (2.21) yields

$$f'(x) = \frac{2x[74x + 108x^3 + (7 + 40x^2)\sqrt{1 + x^2}\sinh^{-1}(x)]}{\sqrt{1 + x^2}} > 0 \tag{2.23}$$

for $x \in (0, 1)$.

Therefore, $\phi_{5/9}(x) < 0$ for all $x \in (0, 1)$ follows easily from (2.19) and (2.20) together with (2.22) and (2.23).

Case 2 $p = 0$. Then (2.18) yields

$$\frac{\phi_0(x)}{1 - x^2} = x(1 + x^2) - (1 - x^2)\sqrt{1 + x^2}\sinh^{-1}(x) := g(x), \tag{2.24}$$

$$g(0) = 0. \tag{2.25}$$

Differentiating (2.24) we get

$$g'(x) = \frac{x[4x\sqrt{1 + x^2} + (1 + 3x^2)\sinh^{-1}(x)]}{\sqrt{1 + x^2}} > 0 \tag{2.26}$$

for $x \in (0, 1)$

Therefore, $\varphi_0(x) > 0$ for $x \in (0, 1)$ easily from (2.16) and (2.17) together with (2.24)-(2.26).

□

3. Bounds for the Neuman-Sándor Mean

In this section we will deal with problems of finding sharp bounds for the Neuman-Sándor Mean $M(a, b)$ in terms of the geometric combinations of harmonic mean $H(a, b)$ and quadratic mean $Q(a, b)$, geometric mean $G(a, b)$ and quadratic mean $Q(a, b)$, harmonic mean $H(a, b)$ and contra-harmonic mean $C(a, b)$, and geometric mean $G(a, b)$ and contra-harmonic mean $C(a, b)$.

Since $H(a, b)$, $G(a, b)$, $M(a, b)$, $Q(a, b)$ and $C(a, b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a > b$. For the later use we denote $x = (a - b)/(a + b) \in (0, 1)$ and $t = \sinh^{-1}(x) \in (0, t^*)$ with $t^* = \log(1 + \sqrt{2}) = 0.881\cdots$.

Theorem 3.1. *The double inequality*

$$H^\alpha(a, b)Q^{1-\alpha}(a, b) < M(a, b) < H^\beta(a, b)Q^{1-\beta}(a, b) \tag{3.1}$$

holds true for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \geq 2/9$ and $\beta \leq 0$.

Proof. First we take the logarithm of each member of (3.1) and next rearrange terms to obtain

$$\beta < \frac{\log[Q(a, b)] - \log[M(a, b)]}{\log[Q(a, b)] - \log[H(a, b)]} < \alpha. \tag{3.2}$$

Note that

$$\frac{M(a, b)}{A(a, b)} = \frac{x}{\sinh^{-1}(x)}, \quad \frac{H(a, b)}{A(a, b)} = 1 - x^2, \quad \frac{Q(a, b)}{A(a, b)} = \sqrt{1 + x^2}. \tag{3.3}$$

Use of (3.3) followed by a substitution $x = \sinh(t)(0 < t < t^*)$, inequality (3.2) becomes

$$\beta < f(t) < \alpha, \tag{3.4}$$

where

$$f(t) = \frac{\log[\cosh(t)] - \log[\sinh(t)/t]}{\log[\cosh(t)] - \log[1 - \sinh^2(t)]} := \frac{f_1(t)}{f_2(t)}. \tag{3.5}$$

In order to use Lemma 2.1, we consider the following

$$\frac{f_1'(t)}{f_2'(t)} = \frac{[3 - \cosh(2t)][\sinh(2t) - 2t]}{2t\sinh^2(t)[5 + \cosh(2t)]} := \phi(t), \tag{3.6}$$

where $\phi(t)$ is defined as in Lemma 2.3.

It follows from Lemmas 2.1 and 2.3 together with (3.6) that

$$f(t) = \frac{f_1(t)}{f_2(t)} = \frac{f_1(t) - f_1(0^+)}{f_2(t) - f_2(0)}$$

is strictly decreasing on $(0, t^*)$. This in turn implies that

$$\lim_{t\to 0^+} f(t) = \frac{2}{9}, \quad \lim_{t\to t^*} f(t) = 0. \tag{3.7}$$

Making use of (3.7) and the monotonicity of $\phi(t)$ we conclude that in order for the double inequality (3.1) to be valid it is necessary and sufficient that $\alpha \geq 2/9$ and $\beta \leq 0$. □

Theorem 3.2. *The two-sided inequality*

$$G^{\alpha}(a,b)Q^{1-\alpha}(a,b) < M(a,b) < G^{\beta}(a,b)Q^{1-\beta}(a,b) \tag{3.8}$$

holds true for all $a,b>0$ with $a\neq b$ if and only if $\alpha\geq 1/3$ and $\beta\leq 0$.

Proof. We will follows lines introduced in the proof of Theorem 3.1. We take the logarithm of each member of (3.8) and next rearrange terms to get

$$\beta < \frac{\log[Q(a,b)]-\log[M(a,b)]}{\log[Q(a,b)]-\log[G(a,b)]} < \alpha. \tag{3.9}$$

Use of (3.3) and $G(a,b)/A(a,b)=\sqrt{1-x^2}$ followed by a substitution $x=\sinh(t)(0<t<t^*)$, inequality (3.9) is equivalent to

$$\beta < g(t) < \alpha, \tag{3.10}$$

where

$$g(t)=\frac{\log[\cosh(t)]-\log[\sinh(t)/t]}{\log[\cosh(t)]-\log[1-\sinh^2(t)]/2} := \frac{g_1(t)}{g_2(t)}. \tag{3.11}$$

Equation (3.11) leads to

$$\frac{g_1'(t)}{g_2'(t)} = \frac{[3-\cosh(2t)][\sinh(2t)-2t]}{8t\sinh^2(t)} = \frac{\sum_{n=1}^{\infty}[2^{2n+1}(2n+4-2^{2n})/(2n+1)!]t^{2n+1}}{\sum_{n=1}^{\infty}[2^{2n+2}/(2n)!]t^{2n+1}}$$

$$=\frac{\sum_{n=0}^{\infty}[2^{2n+4}(n+3-2^{2n+1})/(2n+3)!]t^{2n}}{\sum_{n=0}^{\infty}[2^{2n+4}/(2n+2)!]t^{2n}} := \frac{\sum_{n=0}^{\infty}a_n't^{2n}}{\sum_{n=0}^{\infty}b_n't^{2n}}, \tag{3.12}$$

$$\frac{a_{n+1}'}{b_{n+1}'}-\frac{a_n'}{b_n'} = -\frac{3+(6n+7)2^{2n+1}}{(2n+3)(2n+5)} < 0 \tag{3.13}$$

for all $n\in\{0,1,2,\cdots\}$.

It follows from Lemmas 2.1(1) and (3.12) together with (3.13) that $g_1'(t)/g_2'(t)$ is strictly decreasing on $(0,t^*)$.

From Lemma 2.1 and (3.11) together with $g_1(0^+)=g_2(0)=0$ and the monotonicity of $g_1'(t)/g_2'(t)$ we clearly see that $g(t)$ is strictly decreasing on $(0,t^*)$.

Therefore, Theorem 3.2 follows from the monotonicity of $g(t)$ and (3.10) together with the fact that

$$\lim_{t\to 0^+} g(t)=\frac{1}{3}, \quad \lim_{t\to t^*} g(t)=0.$$

□

Theorem 3.3. *The following simultaneous inequality*

$$H^{\alpha}(a,b)C^{1-\alpha}(a,b) < M(a,b) < H^{\beta}(a,b)C^{1-\beta}(a,b) \tag{3.14}$$

holds true for all $a,b>0$ with $a\neq b$ if and only if $\alpha\geq 5/12$ and $\beta\leq 0$.

Proof. We take the logarithm of each member of (3.14) and next rearrange terms to get

$$\beta < \frac{\log[C(a,b)]-\log[M(a,b)]}{\log[C(a,b)]-\log[H(a,b)]} < \alpha. \tag{3.15}$$

Use of (3.3) and $C(a,b)/A(a,b)=1+x^2$ followed by a substitution $x=\sinh(t)(0<t<t^*)$, inequality (3.15) becomes

$$\beta < h(t) < \alpha, \tag{3.16}$$

where

$$h(t)=\frac{\log[\cosh(t)]-\log[\sinh(t)/t]/2}{\log[\cosh(t)]-\log[1-\sinh^2(t)]/2} := \frac{h_1(t)}{h_2(t)}. \tag{3.17}$$

Equation (3.17) gives

$$
\begin{aligned}
\frac{h_1'(t)}{h_2'(t)} =& \frac{[3-\cosh(2t)][\sinh(2t)+t\cosh(2t)-3t]}{16t\sinh^2(t)}\\
=& \frac{\sum\limits_{n=0}^{\infty}\left[2^{2n+3}\left((3-2^{2n})(2n+3)+3-2^{2n+2}\right)/(2n+3)!\right]t^{2n}}{\sum\limits_{n=0}^{\infty}[2^{2n+5}/(2n+2)!]t^{2n}} := \frac{\sum\limits_{n=0}^{\infty}c_n' t^{2n}}{\sum\limits_{n=0}^{\infty}d_n' t^{2n}}, \qquad (3.18)
\end{aligned}
$$

$$
\frac{c_{n+1}'}{d_{n+1}'} - \frac{c_n'}{d_n'} = -3\times 2^{2n-2} - \frac{3}{2(2n+3)(2n+5)} - \frac{(6n+7)2^{2n}}{(2n+3)(2n+5)} < 0 \qquad (3.19)
$$

for all $n \in \{0, 1, 2, \cdots\}$.

It follows from Lemmas 2.2(1) and (3.18) together with (3.19) that $h_1'(t)/h_2'(t)$ is strictly decreasing on $(0, t^*)$.

From Lemma 2.1 and (3.17) together with $h_1(0^+) = h_2(0) = 0$ and the monotonicity of $h_1'(t)/h_2'(t)$ we clearly see that $h(t)$ is strictly decreasing on $(0, t^*)$.

Therefore, Theorem 3.3 follows from the monotonicity of $h(t)$ and (3.16) together with the fact that

$$
\lim_{t\to 0^+} h(t) = \frac{5}{12}, \quad \lim_{t\to t^*} h(t) = 0.
$$

□

Theorem 3.4. *The following inequality*

$$
G^{\alpha}(a,b)C^{1-\alpha}(a,b) < M(a,b) < G^{\beta}(a,b)C^{1-\beta}(a,b) \qquad (3.20)
$$

is valid for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \geq 5/9$ and $\beta \leq 0$.

Proof. Making use of (3.3) and $C(a,b)/A(a,b) = 1+x^2$ together with $G(a,b)/A(a,b) = \sqrt{1-x^2}$ we get

$$
\frac{\log[C(a,b)] - \log[M(a,b)]}{\log[C(a,b)] - \log[G(a,b)]} = \frac{\log(1+x^2) - \log[x/\sinh^{-1}(x)]}{\log(1+x^2) - \log\sqrt{1-x^2}}. \qquad (3.21)
$$

Elaborated computations lead to

$$
\lim_{x\to 0^+} \frac{\log(1+x^2) - \log[x/\sinh^{-1}(x)]}{\log(1+x^2) - \log\sqrt{1-x^2}} = \frac{5}{9}, \qquad (3.22)
$$

$$
\lim_{x\to 1^-} \frac{\log(1+x^2) - \log[x/\sinh^{-1}(x)]}{\log(1+x^2) - \log\sqrt{1-x^2}} = 0. \qquad (3.23)
$$

Taking the logarithm of (3.20), we consider the difference between the convex combination of $\log G(a,b)$, $\log C(a,b)$ and $\log M(a,b)$ as follows

$$
\begin{aligned}
& p\log G(a,b) + (1-p)\log C(a,b) - \log M(a,b)\\
=& p\log\sqrt{1-x^2} + (1-p)\log(1+x^2) - \log\frac{x}{\sinh^{-1}(x)} = \varphi_p(x), \qquad (3.24)
\end{aligned}
$$

where $\varphi_p(x)$ is defined as in Lemma 2.4.

Therefore, $G^{5/9}(a,b)C^{4/9}(a,b) < M(a,b) < C(a,b)$ for all $a, b > 0$ with $a \neq b$ follows from (3.24) and Lemma 2.4. This in conjunction with the following statements gives the asserted result.

- If $\alpha < 5/9$, then equations (3.21) and (3.22) lead to the conclusion that there exists $0 < \delta_1 < 1$ such that $M(a,b) < G^{\alpha}(a,b)C^{1-\alpha}(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (0, \delta_1)$.
- If $\beta > 0$, then equations (3.21) and (3.23) imply that there exists $0 < \delta_2 < 1$ such that $M(a,b) > G^{\beta}(a,b)C^{1-\beta}(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (1-\delta_2, 1)$.

□

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